Ansatz-independent solution of a soliton in a strong dispersion-management system

A. Ferrando and M. Zacarés
Departament d'Òptica, Universitat de València, E-46100 Burjassot, València, Spain

P. Fernández de Córdoba
Departament de Matemàtica Aplicada, Universitat Politècnica de València, E-46071 València, Spain

(Received 12 May 1999; revised manuscript received 11 April 2000)

We introduce a theoretical approach to the study of propagation in systems with periodic strong-management dispersion. Our approach does not assume any ansatz about the form of the solution nor does it make use of any average procedure. We find an explicit solution for the pulse evolution in the fast dynamics regime (distances smaller than the dispersion period). We also establish the equation of motion governing the slow dynamics of an arbitrary pulse and prove that the pulse evolution is nonlinear and Hamiltonian. We solve this equation and find that a nonlinear solitonlike solution occurs self-consistently in the form of an asymptotic stationary eigenfunction of the Hamiltonian.

PACS number(s): 42.65.Tg, 41.20.Jb

I. INTRODUCTION

A lot of effort has been devoted recently to the understanding of nonlinear electromagnetic propagation in dispersion-management (DM) systems. The experimental feasibility of this kind of propagation has been demonstrated in stretched-pulse lasers [1] and optical communication systems [2], thus drawing considerable interest for its technical advantages with respect to traditional soliton transmission techniques. Pulse evolution presents a very peculiar behavior in DM systems where dispersion is periodic and changes its sign along the fiber. Like media with constant negative dispersion, where stationary solitons were known to exist long ago [3], DM systems seem to possess solitonlike solutions that behave as stationary pulses at long distances. However, they evolve very differently within one dispersion period, where they experiment a severe compression-broadening process not present in ordinary solitons. Although a lot of work has been done to provide a suitable mathematical description of this new type of evolution, most of it relies on the assumption of an ansatz, or a specific property, about the solution that is sought. Variational methods are intrinsically built on a trial function for the solution [4–11]. Average dynamics methods provide evolution equations for averaged quantities, which can be solved under certain assumptions about the form of the solution [12–15]. Another approaches resort to numerical analysis [16] or to truncated modal evolution equations [17].

Using an adapted Green-function formalism for DM systems, we can give a description of the pulse evolution at any position inside one dispersion period (fast dynamics) without using any ansatz or assuming any specific property. Our method is especially well adapted to study strong DM systems since it incorporates a perturbative treatment of the small parameter controlling strong DM effects. In addition, we derive the equation that provides the global evolution of a pulse after many dispersion periods (slow dynamics) without considering any average or truncation procedure. This equation is exact in first-order perturbation theory. Finally, we solve this equation and find that a nonlinear solitonlike solution occurs self-consistently in the form of an asymptotic stationary eigenfunction of the Hamiltonian.

II. PROPER LENGTH METHOD FOR FAST DYNAMICS EVOLUTION

Our starting point is the periodic dispersion nonlinear Schrödinger equation,

$$\frac{\partial u}{\partial z} = -i\frac{D(z)}{2}\frac{\partial^2 u}{\partial t^2} + i\delta|u|^2 u, \tag{1}$$

where $D(z)$ is the periodic dispersion and δ is the nonlinear coefficient.

A linear fiber with homogeneous dispersion is a particular case of Eq. (1), where $D(z)=D$ and $\delta=0$. In such a case, the differential equation is identical to a Schrödinger equation for a one-dimensional (1D) free particle where z plays the role of time and t the role of position. It is well known that a suitable description of the wave-function evolution can be given by introducing the unitary evolution operator $U(t)=\exp(-iHt)$, where H is the free Hamiltonian ($H=-1/2m\,\partial^2/\partial x^2$). When applied to our case of interest, this means that the vector representing the field at position z can be written as

$$|u(z)\rangle = e^{-iHz}|u(0)\rangle \Leftrightarrow \frac{\partial u}{\partial z} = -i\frac{D}{2}\frac{\partial^2 u}{\partial t^2}, \tag{2}$$

where $H=-(D/2)(\partial^2/\partial t^2)$.

We observe that a necessary condition for the above relation to be true is that the Hamiltonian is time-independent (z-independent, in our case). If the fiber dispersion varies with z, then $H \to H(z)$ and the previous relation is no longer true. However, let us imagine an alternative evolution problem where the evolution parameter is given by a different variable, τ, that we will call *proper length* for reasons that will become apparent soon. By definition, this evolution problem corresponds to a homogeneous fiber with $D=1$.

Thus, following the previous equivalence expressed in Eq. (2), the pulse field for this problem, now $u(\tau)$, will satisfy

$$|u(\tau)\rangle = e^{-iH\tau}|u(0)\rangle \Leftrightarrow \frac{\partial u}{\partial \tau} = -i\frac{1}{2}\frac{\partial^2 u}{\partial t^2}. \tag{3}$$

Up to now, both problems are unrelated. At this point, let us assume that τ is a function of z, $\tau = \tau(z)$. Then, Eq. (3) still holds.

$$|u(\tau(z))\rangle = e^{-iH\tau(z)}|u(0)\rangle \Leftrightarrow \frac{\partial u}{\partial \tau(z)} = -i\frac{1}{2}\frac{\partial^2 u}{\partial t^2}. \tag{4}$$

It is clear that, in this case, Eq. (2) appears as a particular case of the above equation when $\tau(z) = Dz$. In the most general case, however, the homogeneous problem in the proper length τ maps into a different z-evolution problem. This is easily checked by performing the change of variable $(\tau \to z \Rightarrow \partial/\partial\tau(z) \to [1/\tau'(z)](\partial/\partial z))$. The differential equation satisfied by the pulse field in the z variable, $\tilde{u}(z) = u(\tau(z))$, is thus

$$\frac{\partial \tilde{u}}{\partial z} = -i\frac{\tau'(z)}{2}\frac{\partial^2 \tilde{u}}{\partial t^2}. \tag{5}$$

A direct comparison with Eq. (1) for the linear case reveals the following crucial relation:

$$\tau'(z) = D(z) \Leftrightarrow \tau(z) = \int_{z_0}^{z} dz' D(z'). \tag{6}$$

This property shows that any inhomogeneous linear problem, given by a z-dependent dispersion function $D(z)$, is equivalent to a homogeneous problem with dispersion $D = 1$ in the proper length function $\tau(z)$, the relation between both being given by Eq. (6). In the general case, the proper length τ has no length dimensions, $[\tau] = T^2$. However, since it plays the same role as z in the wave equation, we will keep this name for it. The proper length has a clear physical meaning. According to the integrated expression in Eq. (6), it represents the accumulated dispersion over the distance interval under consideration. However, here, in its condition of evolution variable, τ plays a different role, which turns out to be very particular in DM systems. In a DM system, where the sign of dispersion changes along one period, the proper length τ, unlike the original parameter z, evolves *differently* in the positive and negative group velocity dispersion (GVD) regions $[d\tau/dz = D(z)]$. There is a forward (ordinary) evolution in the positive GVD regions, whereas evolution is reversed (backward) in the negative GVD sections. For this reason, one finds the typical breathing behavior of pulse evolution even in linear DM systems, when $g = 0$ (see [17], for instance).

Now, we consider the general case of propagation in a nonlinear medium with inhomogeneous dispersion, described by Eq. (1). We have learned how to deal with the most general case of z-dependent dispersion in the linear case. In the nonlinear case, it is also very interesting to introduce the concept of proper length. The way of introducing the proper length function $\tau(z)$ in Eq. (1) is simple. We divide both sides of Eq. (1) by $D(z)$, use the equivalence

$$\frac{1}{D(z)}\frac{\partial u}{\partial z} = \frac{1}{\tau'(z)}\frac{\partial u}{\partial z} = \frac{\partial u}{\partial \tau}, \tag{7}$$

and introduce the τ-dependent ratio $g(\tau) \equiv \delta/D(\tau)$ [since D is z-dependent, it also depends on τ through the inverse relation $z = \tau^{-1}(\tau)$]. This equation becomes, in the τ variable,

$$\frac{\partial u}{\partial \tau} = -i\frac{1}{2}\frac{\partial^2 u}{\partial t^2} + ig(\tau)|u|^2 u, \tag{8}$$

where, for convenience, we are using the same notation for $u(\tau)$ and $u(z)$ provided there is no confusion [before we have distinguished between them: $\tilde{u}(z) = u(\tau(z))$]. The τ-dependent function $g(\tau)$ acts now as an inhomogeneous nonlinear coupling and has dimensions of T^{-2}. It is convenient to use the dimensionless time variables $\bar{t} = t/t_0$, $\bar{\tau} = \tau/t_0^2$, and $\bar{g} = g t_0^2$, where t_0 is a typical time scale of the problem, e.g., the initial pulse width. Equation (8) remains exactly the same but with τ, t, and g substituted by their normalized counterparts. For this reason, and from now on, we will consider all time variables properly normalized, although we keep using the unbarred notation.

The normalized coupling constant has an interesting physical meaning. Its absolute value is merely the ratio between the dispersion and nonlinear lengths L_D/L_{NL}, defined as

$$L_D = \frac{t_0^2}{|D|} \quad \text{and} \quad L_{\mathrm{NL}} = \frac{1}{\delta}. \tag{9}$$

The dispersion length L_D and the nonlinear length L_{NL} provide the length over which the dispersion and nonlinear effects become important for pulse evolution along a fiber of length L. By comparing these three lengths, we can determine which effects are relevant or not for pulse propagation. We will assume here that nonlinear effects are less important than dispersion ones, so that $L_D \approx L$ and $L_{\mathrm{NL}} \gg L_D$, and therefore the effective coupling function $g(\tau)$ will be small. Notice that the coupling function $g(\tau)$ is not constant but position-dependent and it will have to be treated carefully. In order to introduce a real constant parameter, let g be the maximum value that the L_D/L_{NL} ratio can reach over one dispersion period. Then $|g(\tau)| \leqslant g \forall \tau$, and we can write the coupling constant function as $g(\tau) = g l(\tau)$, where $|l(\tau)| \leqslant 1 \forall \tau$. In this way, we can give a rigorous perturbative approach in terms of the now real coupling constant g and calculate, in principle, arbitrary order corrections to the pure dispersion pulse solution.

We are interested in finding a solution of Eq. (8), $u(t,\tau;g)$, that exists in the limit $g \to 0$. We will assume that a small nonlinearity produces small perturbations in the pulse profile, so that the amplitude can be expanded in a power series in g,

$$u(t,\tau;g) = u_0(t,\tau) + g u_1(t,\tau) + g^2 u_2(t,\tau) + \cdots. \tag{10}$$

We plug now the previous expansion in both sides of Eq. (8) and equal coefficients of the same order in g.

To lowest order

$$\frac{\partial u_0}{\partial \tau} = -i\frac{1}{2}\frac{\partial^2 u_0}{\partial t^2}. \tag{11}$$

To order g

$$\frac{\partial u_1}{\partial \tau} = -i\frac{1}{2}\frac{\partial^2 u_1}{\partial t^2} + \rho_0(t,\tau), \tag{12}$$

where $\rho_0(t,\tau) = il(\tau)|u_0|^2 u_0$ acts as a density in an active medium.

In general,

$$\frac{\partial u_n}{\partial \tau} = -i\frac{1}{2}\frac{\partial^2 u_n}{\partial t^2} + \rho_{n-1}(t,\tau), \tag{13}$$

where the $(n-1)$th order "density," ρ_{n-1}, can be constructed systematically from previously known lower order solutions.

In this way, we have transformed the original nonlinear equation in a set of linear differential equations that can be solved recursively. All the nonlinear corrections verify inhomogeneous linear equations, with source terms, that involve the same differential operator, $\partial/\partial\tau + (i/2)\partial^2/\partial t^2$. All of them can be solved by means of the Green-function method. The associated Green problem to be solved first is

$$\left(\frac{\partial}{\partial \tau} + \frac{i}{2}\frac{\partial^2}{\partial t^2}\right) G(t-t',\tau-\tau') = \delta(t-t')\,\delta(\tau-\tau'). \tag{14}$$

According to the Green-function method, the solution for the field has to be of the form

$$u_{\mathrm{NL}}(t,\tau) = \int dt' \int_{\tau_{\min}}^{\tau_{\max}} d\tau'\, G(t-t',\tau-\tau')\,\rho(t',\tau'), \tag{15}$$

where $\tau_{\min}$ and $\tau_{\max}$ are the minimum and maximum, respectively, of the interval where the function τ is defined, and ρ is a generic source density function. However, we have to be very careful in defining the integration domain in the τ variable. As we have already seen, evolution in z is unidirectional whereas evolution in τ is forward or backward depending on the sign of the dispersion. Therefore, the integration domain is ambiguous in τ, although it is not in the z variable. Consequently, in order to deal with the integration limits of Eq. (15) properly, we transform the integral in τ' in an integral in z'. The result is

$$u_{\mathrm{NL}}(t,z) = \int dt' \int_{-L/2}^{L/2} dz'\, \tilde{G}_+(t,t';z,z')\,\tilde{\rho}(t',z'), \tag{16}$$

where $\tilde{\rho} = D(z)\rho$, and $\tilde{G}_+$ represents the advanced Green function in the z variable, satisfying

$$\left(\frac{\partial}{\partial z} + \frac{iD(z)}{2}\frac{\partial^2}{\partial t^2}\right)\tilde{G}_+(t-t',z-z') = \delta(t-t')\,\delta(z-z') \tag{17}$$

and the advanced condition $\tilde{G}_+ \sim \theta(z-z')$ [$\theta()$ being the Heaviside function]. The relation between $\tilde{G}_+$ and the τ Green function G depends on the sign of the dispersion. This is so because the $z \to \tau$ mapping is not one-to-one and, therefore, the change of variable becomes ambiguous. In order to avoid this ambiguity, we divide the $[-L/2,L/2]$ interval in smaller subintervals Δ_i of alternating dispersion D_i. In each of these subintervals with a well-defined sign of the dispersion, the $z\to\tau$ mapping is one-to-one, and the ambiguity disappears. We choose now a generic subinterval Δ_i to perform the change of variable. Dividing Eq. (17) by $D_i(z) = d\tau(z)/dz, z\in\Delta_i$, introducing the proper length function $\tau(z)$, and taking into account the property of the δ function,

$$\begin{aligned}\delta(\tau(z)-\tau(z')) &= \frac{1}{|D_i(z)|}\,\delta(z-z')\\ &= \frac{1}{\mathrm{sgn}(D_i)D_i(z)}\,\delta(z-z'),\end{aligned} \tag{18}$$

we obtain the equation for the Green function in the τ variable for positive and negative dispersion subintervals,

$$\left(\frac{\partial}{\partial \tau} + \frac{i}{2}\frac{\partial^2}{\partial t^2}\right) G_i(t-t',\tau-\tau') = \mathrm{sgn}(D_i)\,\delta(t-t')\,\delta(\tau-\tau'), \tag{19}$$

where $\tau = \tau(z)$ and $\tau' = \tau(z')$. Notice that in the positive GVD sectors we obtain the standard Green-function equation for an ordinary forward τ propagation, $G \sim \theta(\tau-\tau')$. However, for negative GVD we obtain an extra minus sign in the right-hand side of Eq. (19). This is a consequence of backward τ propagation, $G \sim \theta(\tau'-\tau)$, because then the τ derivative produces an extra sign, $[d\theta(\tau'-\tau)]/d\tau = -\delta(\tau-\tau')$.

The solution of this Green-function problem is well known. In the positive GVD intervals, we have

$$\begin{aligned}\tilde{G}_+(t,t';z,z') &= G_+(t-t',\tau-\tau')\\ &= \theta(\tau-\tau')G_0(t-t',\tau-\tau'),\\ \text{where} \quad \tau &= \tau(z), \quad \tau' = \tau(z'),\end{aligned} \tag{20}$$

whereas for the negative GVD sections,

$$\begin{aligned}\tilde{G}_+(t,t';z,z') &= G_-(t-t',\tau-\tau')\\ &= \theta(\tau'-\tau)G_0(t-t',\tau-\tau'),\\ \text{where} \quad \tau &= \tau(z), \tau' = \tau(z').\end{aligned} \tag{21}$$

Notice that these equations show, as pointed out before, how τ evolution occurs in the same direction as z evolution in the positive GVD intervals ($\tau > \tau'$, when $z > z'$), while just the opposite occurs in the negative GVD sections ($\tau < \tau'$, when $z > z'$). The form of $\tilde{G}_+$ is completely determined by the above expressions, because the function G_0 is merely the representation of the unitary evolution operator in the time domain,

$$G_0(t-t',\tau-\tau')=\langle t|e^{-iH(\tau-\tau')}|t'\rangle = e^{i(\pi/4)}\left(\frac{1}{2\pi(\tau-\tau')}\right)^{1/2} e^{-i[(t-t')^2/2(\tau-\tau')]}. \tag{22}$$

These properties allow us to write a closed expression for $\widetilde{G}_+$,

$$\widetilde{G}_+(t,t';z,z')=\theta(z-z')G_0(t-t',\tau(z)-\tau(z')). \tag{23}$$

Now we split up the integral (16) in a sum over the partition $[-L/2,L/2]=\cup_{i=1}^{N}\Delta_i$, where $\Delta_i=[z_{i-1},z_i]$ (we assume that $z\in\Delta_n$, $1\leqslant n\leqslant N$),

$$\begin{aligned} u_{\rm NL}(t,z)&=\sum_{i=1}^{N}\int dt'\int_{\Delta_i}dz'\,\theta(z-z')\\ &\quad\times G_0(t-t',\tau(z)-\tau(z'))\widetilde{\rho}(t',z')\\ &=\sum_{i=1}^{n-1}\int dt'\int_{z_{i-1}}^{z_i}dz'\,G_0(t-t',\tau(z)-\tau(z'))\\ &\quad\times\widetilde{\rho}(t',z')+\int dt'\int_{z_{n-1}}^{z}dz'\,G_0(t-t',\tau(z)\\ &\quad-\tau(z'))\widetilde{\rho}(t',z'). \end{aligned} \tag{24}$$

The previous equation solves the ambiguity in the definition of the integration domain of the Green formulation in the τ variable. The inverse of $\tau(z')$ is now a true single-valued function in each Δ_i. Under these conditions, the change of variables $z'\to\tau'$ is now allowed in each of the integrals appearing in the sum over Δ_i's in Eq. (24). The Jacobian of the transformation, $[d\tau(z')/dz']^{-1}$, is precisely the inverse of the dispersion function in Δ_i. The result is (recall that $\widetilde{\rho}=D\rho$)

$$\begin{aligned} u_{\rm NL}(t,\tau)&=\sum_{i=1}^{n-1}\int dt'\int_{\tau_{i-1}}^{\tau_i}d\tau'\,G_0(t-t',\tau-\tau')\rho_i(t',\tau')\\ &\quad+\int dt'\int_{\tau_{n-1}}^{\tau}d\tau'\,G_0(t-t',\tau-\tau')\rho_n(t',\tau'). \end{aligned} \tag{25}$$

The above equation gives the most general answer to the problem of finding any arbitrary order nonlinear correction once the density ρ has been constructed out of lower order contributions. At the same time, it incorporates the effects of the inhomogeneity and the sign change in the dispersion. Using recursively this equation and the equations giving the different order terms of the density ρ, we could solve perturbatively, to any order in the coupling constant g, the problem of pulse propagation in a nonlinear medium with inhomogeneous and sign-changing dispersion. It is clear that, although the formalism is completely general, realistic calculations of high order terms can be extremely involved. In any case, it is of great interest to have a completely general expression that can be used to demonstrate general properties, valid to all orders in perturbation theory, that will affect any solution for which a series expansion can apply.

III. ANALYTICAL RESULTS FOR THE PROPAGATION OF A GAUSSIAN PULSE

In the first part of this section, we will find the leading term of the perturbative expansion (10), that is, the linear term u_0. This is given by the operator relation Eq. (4) expressed in the time domain as follows:

$$u_0(t,\tau(z))=\int dt'\,G_0(t-t',\tau(z))u_0(t',0), \tag{26}$$

where

$$G_0(t-t',\tau(z))=\langle t|e^{-iH\tau(z)}|t'\rangle = e^{i(\pi/4)}\left(\frac{1}{2\pi\tau(z)}\right)^{1/2} e^{-i[(t-t')^2/2\tau(z)]}. \tag{27}$$

This relation is valid for any propagating pulse shape. An important case of propagation is that of a Gaussian pulse. Let us assume that at the initial fiber position z_0 [$\tau(z_0)=0$, according to Eq. (6)], the pulse has the Gaussian form

$$u_0(t,0)=e^{-t^2/2t_0^2}, \tag{28}$$

where t_0 is the pulse width. Then, a straightforward use of the evolution integral (26) yields the well-known expression for the field at position z [17],

$$u_0(t,\tau(z))=\frac{t_0}{\sqrt{t_0^2-i\tau(z)}}e^{-t^2/\{2[t_0^2-i\tau(z)]\}}. \tag{29}$$

In fiber ring lasers or in dispersion managed communication systems, dispersion changes in the different sections the fiber is made of, but since each section is fabricated with the same material, dispersion is constant in each piece. In what follows, we will restrict ourselves to a simple case of a DM system, namely, that corresponding to zero average periodic dispersion $D(z+L)=D(z)$ in a symmetric dispersion map system of length L,

$$D(z)=\begin{cases}-d, & z\in[-L/2,0[,\\ +d, & z\in[0,L/2[.\end{cases} \tag{30}$$

Our next step will be to find the first nonlinear perturbative correction to the problem of the fiber given above. That is, we want to evaluate the integral (25) for this case, where $\tau(z)=d(|z|-L/2)$, and $\rho_0=il|u_0|^2u_0$,

$$u_1(t,z) = -i\int dt'\int_0^{\tau(z)} d\tau' G_0(t-t',\tau(z)-\tau') \times |u_0(t',\tau')|^2 u_0(t',\tau') + \theta(z)2i \times \int dt'\int_{\tau_1}^{\tau(z)} d\tau' G_0(t-t',\tau(z)-\tau') \times |u_0(t',\tau')|^2 u_0(t',\tau'), \quad (31)$$

where $\tau_1 = -dL/(2t_0^2)$ in this case.

Now we introduce the lowest order solution, $u_0(t,\tau)$, from Eq. (29), into Eq. (31). The integration over the time variable can be done because it has the form of the linear evolution of a Gaussian pulse. Consequently, the first-order nonlinear contribution is the result of an integral over the intermediate proper length variable τ'. If we set $t_0 = 1$, so that all time-dependent magnitudes will be immediately normalized to the original Gaussian pulse width t_0, we can write the following expressions for the linear and the first nonlinear correction (the dependence of τ on z is always assumed although it is not written explicitly):

$$u_0(t,z) = \frac{1}{\sqrt{1-i\tau}} \exp\left[-\frac{t^2}{2}\frac{1}{(1-i\tau)}\right] \quad (32)$$

and

$$u_1(t,z) = -i\int_0^{\tau} d\tau' \frac{1}{\sqrt{(1-i\tau')\Delta(\tau,\tau')}} \times \exp\left[-\frac{t^2}{2}\left(\frac{3+i\tau'}{\Delta(\tau,\tau')}\right)\right] + \theta(z)2i\int_{\tau_1}^{\tau} d\tau' \times \frac{1}{\sqrt{(1-i\tau')\Delta(\tau,\tau')}} \exp\left[-\frac{t^2}{2}\left(\frac{3+i\tau'}{\Delta(\tau,\tau')}\right)\right], \quad (33)$$

where $\Delta(\tau,\tau') = 1 + 3i\tau' + \tau(\tau' - 3i)$.

Equation (33) gives the value of the first nonlinear correction at an arbitrary point of the fiber z. It is also of great interest to know the net effect of the nonlinear perturbation after the Gaussian pulse has completed an entire round. This contribution is the value of u_1 at the point $z = L/2$. The associated proper length is zero, which makes this contribution easier to handle,

$$u_1\left(t,\frac{L}{2}\right) = 2i\int_{\tau_1}^{0} d\tau' \left(\frac{1}{1+2i\tau'+3\tau'^2}\right)^{1/2} \times \exp\left[-\frac{t^2(\tau'-3i)}{6\tau'-2i}\right]. \quad (34)$$

In order to check the validity of the analytical expression for the first-order correction, we have performed an alternative numerical simulation of the Gaussian pulse evolution in a dispersion managed system with dispersion map given by Eq. (30) within one period. This simulation is based on an standard split-step Fourier procedure [19]. On the one hand, we calculate the first nonlinear amplitude u_1 using the analytical equation (33). On the other hand, we evaluate numerically the complete pulse profile amplitude $u(t,z,g)$ through our simulation code. We obtain the full nonlinear contribution within one period by substracting the linear part u_0 from the previously numerically evaluated function u. Now we can compare directly the first nonlinear correction gu_1 with this full nonlinear contribution. According to the perturbative expansion (10), they should agree up to order g^2. In Fig. 1, we represent the error in the real and imaginary parts of the functional difference $(u - u_0 - gu_1)$ divided by g^2 as a function of g, where the error is calculated as the maximum of the absolute value of these differences in both the time and $z \in [-L/2, L/2]$ domains. In order to check the consistency of the expansion, we plot these errors for two different values of τ_1. In all cases, the error curves are consistent with the $O(g^2)$ approximation. This check shows simultaneously the precision achieved in the analytical calculation and the consistency of the proper length Green-function method.

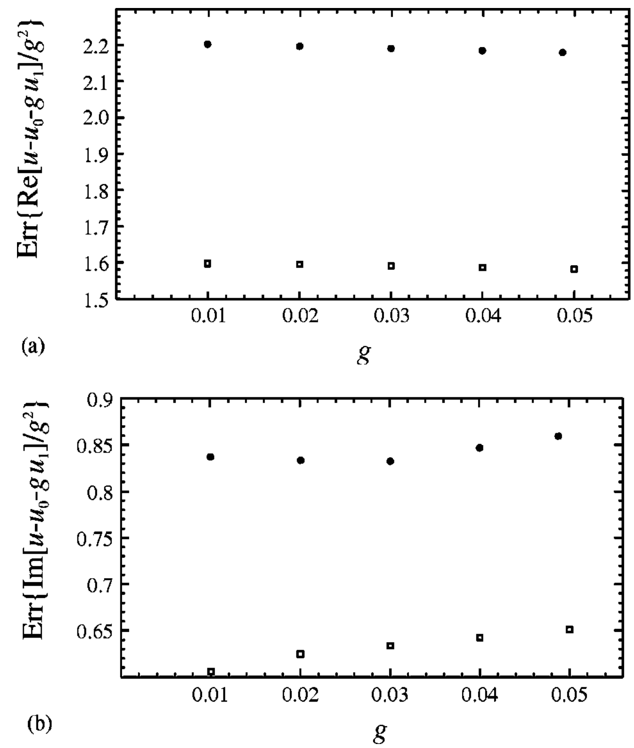

FIG. 1. Numerical consistency check of the analytical solution for evolution within one period. (a) Error in the real part of the functional difference $(u - u_0 - gu_1)/g^2$ as a function of the coupling constant g. (b) Same for the imaginary part. We plot these errors for $\tau_1 = -1.5$ (squares) and $\tau_1 = -2$ (circles).

We also present in Fig. 2 and Fig. 3 the z evolution for both the total and the first nonlinear pulse amplitudes, u and u_1, respectively. Figure 2 has been evaluated using the split-step Fourier numerical method, whereas Fig. 3 is obtained from the analytical expression (33). Notice, however, that according to the accuracy of the expansion, represented in Fig. 1, the latter plot would not be distinguishable from that of the full nonlinear correction provided by the numerical simulation. For the same reason, the analytical perturbative result for u would provide the same plot as the numerical method used in Fig. 2.

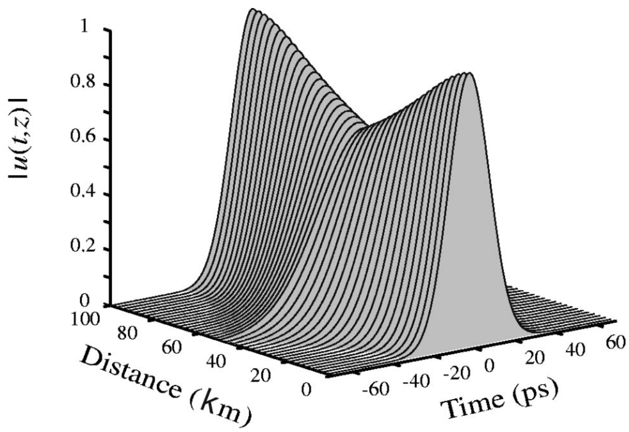

FIG. 2. z evolution, within one dispersion period, of the modulus of the total amplitude, $|u|$, for a Gaussian pulse with $\tau_1 = -2$ and $g = 0.04$.

IV. EXACT DERIVATION OF THE EQUATION OF MOTION IN THE SLOW DYNAMICS REGIME

If we are interested in the behavior of the pulse amplitude always at a particular reference plane of the period of the fiber, say $z = -L/2 + jL, j \in Z$ (the beginning of the negative dispersion section), we need only to know the values of u_0 and u_1 at $\tau(L/2) = 0$. In such a case, the first summand in Eq. (33) vanishes and the general expression simplifies. A Gaussian pulse modifies its amplitude and phase even in a single round, as is apparent from the analytical result given in Eq. (34). A similar result is obtained for a pulse described by an arbitrary Hermite-Gauss function. In general, an arbitrary pulse written as a linear combination of the normalized Hermite-Gauss basis, $u(t,\tau) = \Sigma_n \lambda_n(\tau) \overline{h}_n(t) e^{-t^2/2}$ ($\int \overline{h}_n \overline{h}_m e^{-t^2} = \delta_{nm}$), will evolve over one dispersion period and will experiment a net variation when a full round is completed ($\tau = 0$). The amount that an arbitrary pulse changes after one dispersion period can be *exactly* evaluated, up to order g^2, because we know how to calculate analytically the variation experimented by the elements of the Hermite-Gauss basis in which the pulse amplitude is expanded. More explicitly, we can evaluate the increment experimented by the components of the pulse amplitude in the Hermite-Gauss basis, the λ vector, up to order g^2. Let $\lambda_n^{(j)}$ be the n component of the pulse vector at the spatial slice $z_j = -L/2 + jL$ characterizing the reference plane j. Then we can obtain the value of this component at the next reference plane at z_{j+1} as follows:

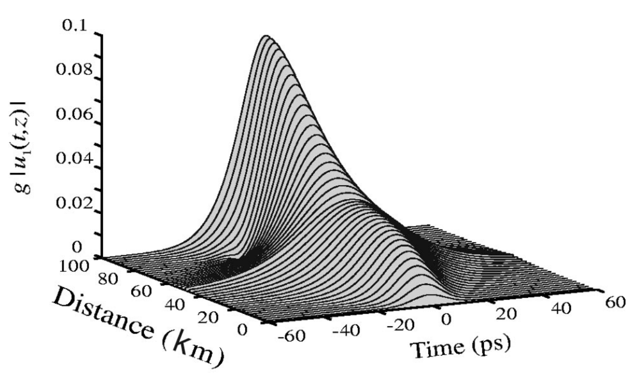

FIG. 3. z evolution, within one dispersion period, of the modulus of the first nonlinear correction, $g|u_1|$, for a Gaussian pulse with $\tau_1 = -2$ and $g = 0.04$.

$$\lambda_n^{(j+1)} = \lambda_n^{(j)} + iH_{nm}^{(j)} \lambda_m^{(j)} + O(g^2),$$

$$H_{nm}^{(j)} = \frac{2g}{L} \lambda_k^{(j)*} T_{nkmp} \lambda_p^{(j)}, \tag{35}$$

where T_{nkmp} is a j-independent fourth-order tensor that depends on τ_1 exclusively. The T tensor can be analytically calculated and its value is given by the following expression:

$$T_{nkmp} = \frac{1}{N_n N_k N_m N_p} \tau_{nkmp} \mathcal{H}_{nkmp}, \tag{36}$$

where the normalization constant N_n is

$$N_n = \sqrt{\sqrt{\pi} 2^n n!} \tag{37}$$

and

$$\tau_{nkmp} = \int_{\tau_1}^{0} d\tau' \frac{1}{\sqrt{1+\tau'^2}} \left(\frac{1+i\tau'}{1-i\tau'} \right)^{[n+k-(m+p)]/2}, \tag{38}$$

$$\mathcal{H}_{nkmp} = \int_{-\infty}^{\infty} dx \prod_{l=n,k,m,p} \overline{h}_l(x) \exp(-2x^2). \tag{39}$$

Using the definition of the incomplete beta function (see [20]) we can write

$$\tau_{nkmp} = i \left\{ B_{(1+i\tau_1)/2} \left(\frac{1+\nu}{2}, \frac{1-\nu}{2} \right) - B_{1/2} \left(\frac{1+\nu}{2}, \frac{1-\nu}{2} \right) \right\}, \tag{40}$$

where

$$\nu = n + k - m - p. \tag{41}$$

The T tensor enjoys nontrivial symmetry properties: it is symmetric under the permutations $n \leftrightarrow k$ and $m \leftrightarrow p$ and, in addition, verifies the self-adjoint-like condition $T_{(mp)(nk)} = T^*_{(nk)(mp)}$. Because of these properties the matrix $H^{(j)}$ is self-adjoint $H^{(j)} = H^{(j)\dagger}$ at all reference planes. This is an important property for the dynamics, because up to $O(g^2)$ terms, Eq. (35) is equivalent to the matrix equation $\lambda^{(j+1)} = \exp(iH^{(j)}L)\lambda^{(j)}$, and thus the self-adjoint character of $H^{(j)}$ gives rise to a unitary evolution operator. The unitary character of the pulse evolution guarantees the conservation of the pulse power $\int |u(t)|^2 = \Sigma_n |\lambda_n|^2$ in first-order perturbation theory. The above equations can be considered as a discretization of a differential equation in a continuous variable Z such that its discretized values coincide with the positions of the reference planes z_j. This is a rather realistic situation because we are interested in the slow dynamics of pulses especially after very long distances, much larger than the dispersion period L. Since the lattice spacing of the discretization is L and it is small over long distances, we can formulate the continuous counterpart of Eq. (35) in the form of a differential master equation for the slow dynamics evolution:

$$-i\frac{d\lambda_n(Z)}{dZ}=H_{nm}(Z)\lambda_m(Z)+O(g^2), \tag{42}$$

where as before $H_{nm}(Z)=2g/L\lambda_k^*(Z)T_{nkmp}\lambda_p(Z)$.

Either in its discrete or continuous form, the evolution equation shows that the slow dynamics of an arbitrary pulse is nonlinear and Hamiltonian, as in ordinary solitons. However, the explicit form of the evolution is different. It is clear that the H matrix operator plays the role of a nonlinear self-adjoint Hamiltonian governing the slow dynamics evolution of the pulse propagation. It is important to stress that no ansatz nor variational approach has been followed to obtain the previous evolution equation. Although from the formal point of view our master equation can have some similarities to an average equation, the way in which we have obtained it differs qualitatively from standard average procedures. The proper length Green-function method permits the determination of the pulse amplitude exactly within one period in first-order perturbation theory for an arbitrary dispersion map, independent of the initial pulse profile. In particular, we can evaluate the pulse amplitude (exactly in this order of perturbation theory) at the end of the dispersion map. This is the information included in the master equation (42), which therefore represents a different approach to the slow dynamics evolution problem from that adopted by averaging the original equation of motion [18].

Although the master equation was derived for a dispersion map such as that presented in Eq. (30), our method is flexible and permits us to incorporate other dispersion maps that would lead to slightly different evolution equations. For example, if the average dispersion is different from zero, one can prove that the nonlinear Hamiltonian operator H is modified by the presence of a linear term. If, besides, the average dispersion is small, as is usual in this kind of system, then $\overline{D}$ can be treated perturbatively to obtain

$$-i\frac{d\lambda_n(Z)}{dZ}=\overline{D}H^0_{nm}\lambda_m(Z)+H^1_{nm}(Z,\overline{D})\lambda_m(Z)+O(g^2,\overline{D}^2), \tag{43}$$

where H^0 is a constant second-order tensor (independent of τ_1 and g) and H^1 is given by

$$H^1_{nm}(Z,\overline{D})=2\frac{g}{L}\lambda_k^*(Z)T_{nkpm}(\overline{D})\lambda_p(Z), \tag{44}$$

where $T(\overline{D})$ includes the first-order correction to the zero average dispersion tensor T.

These equations provide the exact slow dynamics (long distance) evolution of an arbitrary pulse in a strong DM system in first-order perturbation theory.

V. THE DM SOLITON AS A STATIONARY SOLUTION

The question of the existence of soliton solutions can be put forward in an easy way now. A soliton will be a stationary solution of the long-distance evolution equation, which in its continuous form will be given by $\lambda_S(Z)=\alpha_S e^{i\beta_S Z}$. Therefore, it will have to fulfill the stationary matrix eigenvalue equation,

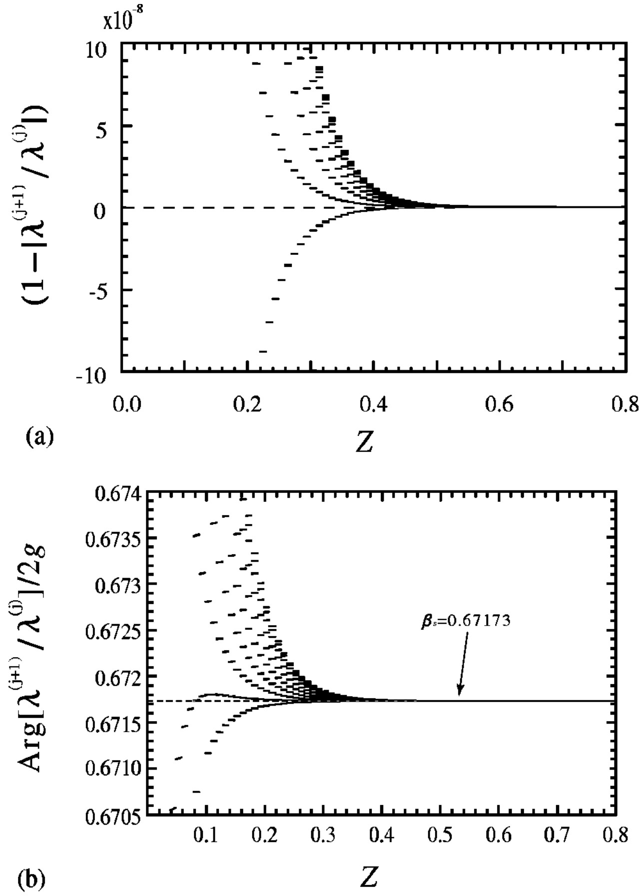

FIG. 4. Axial evolution of the components of the $\lambda^{(j+1)}/\lambda^{(j)}$ ratio for an initial Gaussian pulse using the projection prescription. The normalized axial distance Z is dimensionless and $\tau_1=-2$. The number of Hermite-Gauss functions used is 10. (a) Asymptotic evolution of the modulus to one. (b) Asymptotic evolution of the normalized phase (phase/$2g$) to the stationary propagation constant β_S (horizontal dashed line).

$$H_S\alpha_S=\beta_S\alpha_S, \tag{45}$$

besides the self-consistency condition

$$(H_S)_{nm}=\frac{2g}{L}(\alpha_S^*)_k T_{nkmp}(\alpha_S)_p. \tag{46}$$

In order to preserve the highest degree of generality, we perform the following scale transformations in the evolution equation: $Z\rightarrow(2g/L)Z$ and $H\rightarrow(L/2g)H$ [consequently, $\beta\rightarrow(L/2g)\beta$ as well]. In this way, the evolution equation becomes simultaneously dimensionless and independent of g and L. Only the τ_1 dependence remains through the tensor T.

In order to solve the soliton equations, we will follow a strategy based on physical grounds. Experimental and numerical results indicate that a Gaussian pulse launched in a DM system will evolve to turn into a stationary solution at long distances. Physically, as in ordinary solitons, this evolution corresponds to a radiation process in which the pulse reshapes as it evolves until it reaches its asymptotic stationary form. This radiation has the form of dispersive waves that modify the pulse energy as the input pulse evolves to become the fundamental soliton [19].

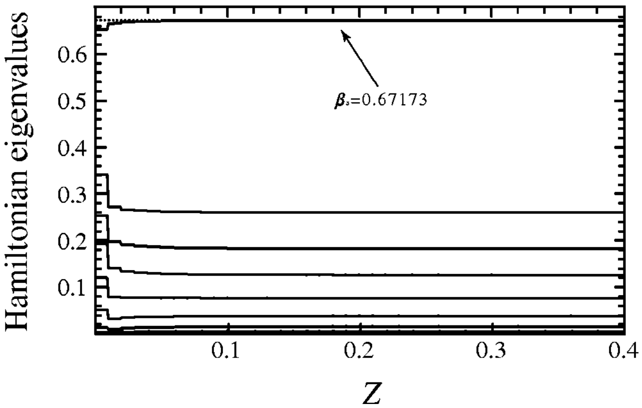

FIG. 5. Axial evolution of the Z-dependent Hamiltonian eigenvalues starting from a Gaussian pulse ($\tau_1=-2$). All eigenvalues tend quickly to their asymptotic values. The highest eigenvalue evolves asymptotically to β_S (horizontal dashed line). This confirms that the asymptotic solution of Fig. 4 satisfies the stationary nonlinear evolution equation.

In what follows, we will face the radiation problem in an alternative way by resorting to an analogy that will allow us to solve the problem of finding the DM soliton in a more efficient manner. Temporal pulse motion in z [given by $u(t,z)$] is analogous to the z evolution of a spatial amplitude $u(x,z)$ in a one-dimensional grad-index medium (a planar waveguide). The refractive index variation of the waveguide is effectively induced by the u field through the nonlinearity, and it is thus dependent on the z coordinate as well. At different z's, due to the local index variation, besides the guided modes, there are radiation modes that correspond to propagating waves with high axial angles. Waves with high axial angles are the first ones to be eliminated by the radiation process. In terms of the propagation constant of these waves, β, the higher angles correspond to the smaller values of β. After some distance, most of the nonradiated power accumulates in the more paraxial modes, i.e., those having the larger values of β. In this way, the projection of the wave amplitude onto the modes with higher propagation constant at different z's effectively eliminates the undesired effects of the dispersive waves. On the other hand, we expect the DM soliton to be the fundamental mode of the equivalent waveguide generated by the nonlinearity. Consequently, the procedure to asymptotically converge to the fundamental mode can be highly accelerated by projecting at different z_j slices the amplitude (as a λ vector) onto the eigenmode of the Hamiltonian $H^{(j)}$ with the highest value of β.

In Fig. 4, we present the Z evolution of the modulus and phase of the different components of the $\lambda^{(j+1)}/\lambda^{(j)}$ ratio calculated using this projection prescription. In the stationary regime this ratio has to be the same for all the components and equals $e^{i\beta_S 2g}$. Although the stationary regime can be only achieved in the strictest sense at infinitely long distances, a fast convergence leads the pulse to a nearly stationary state after not many periods. The same asymptotical behavior is found letting the system evolve freely without the projection prescription. However, stationariness is achieved considerably earlier when this projection prescription is included. Physically, we can understand the projection procedure as a strong and discontinuous elimination of the nonrelevant radiation modes.

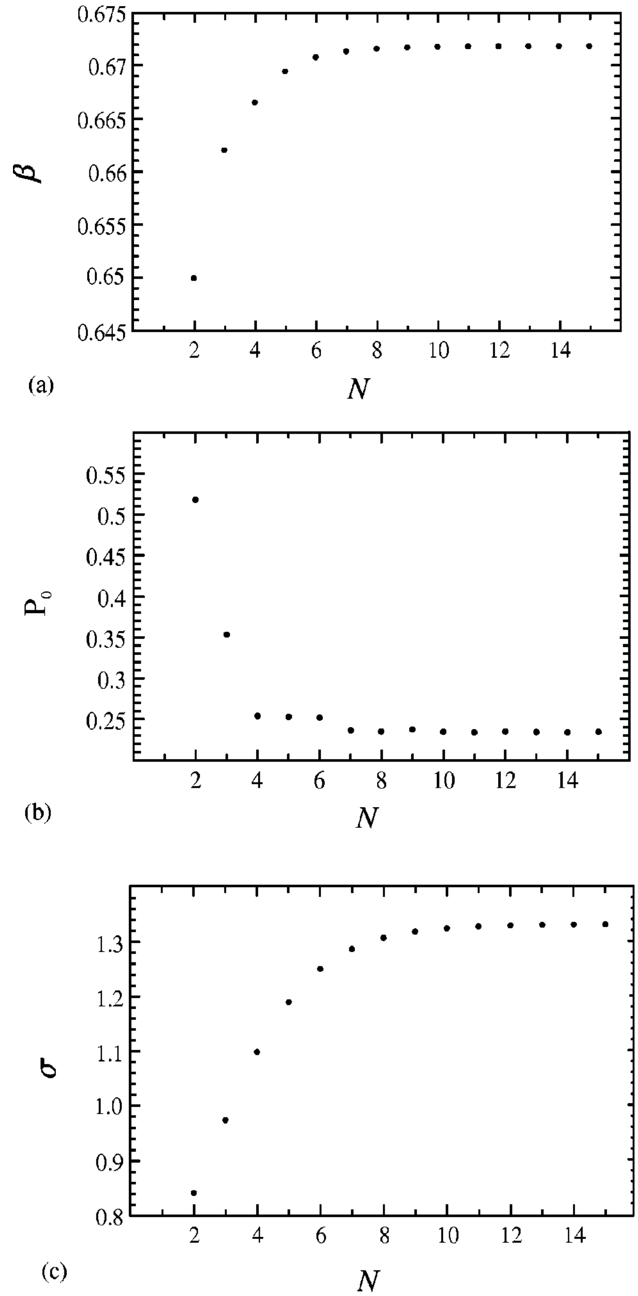

FIG. 6. Convergence of the soliton parameters with the number of modes N ($\tau_1=-2$). (a) Propagation constant, β_s. (b) Peak power, P_0. (c) Root-mean-square width, σ.

We can confirm this way to stationariness in Fig. 4(a), where the modulus of several ratios for different components is shown and the convergence to 1 is apparent, and in Fig. 4(b) where the argument of this ratio tends to stabilize and reach its asymptotic value β_S pretty soon as well (we have normalized the phase to $2g$ in this figure). From this calculation, we can obtain the asymptotic value corresponding to the scaled propagation constant of the stationary solution at a given value of τ_1.

In order to complete the picture, we can verify in a different way that the stationary eigenvalue equation (45) besides the self-consistency condition (46) is satisfied by the asymptotic solution. This can be done by studying the evolution of the eigenvalues of the Hamiltonian as a

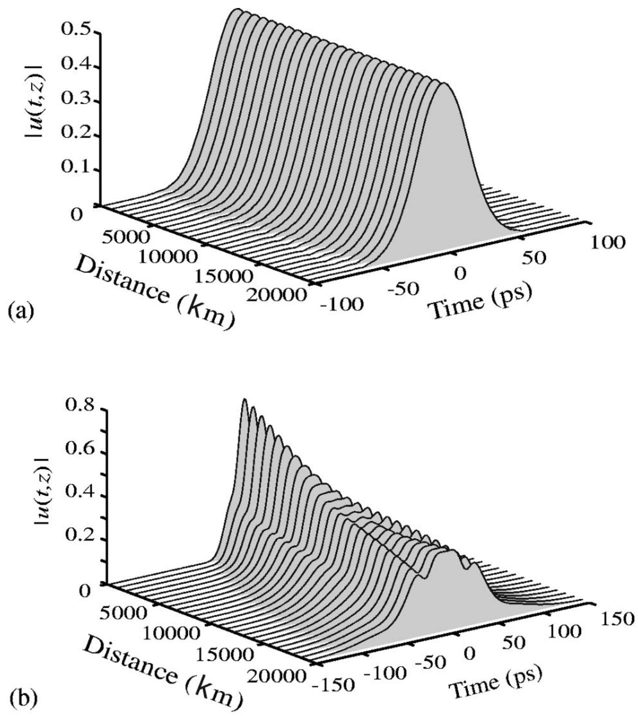

FIG. 7. Long-distance evolution of the master equation solitons: (a) for $N=10$ and (b) for $N=2$. Evolution reaches $Z=20$, which, after properly restoring dimensions, corresponds to a physical distance of $20L/(2g)$. For a standard DM communication system, $g\approx 0.05$ and $L\approx 100$ km, so that the maximum distance would be equivalent to 20 000 km.

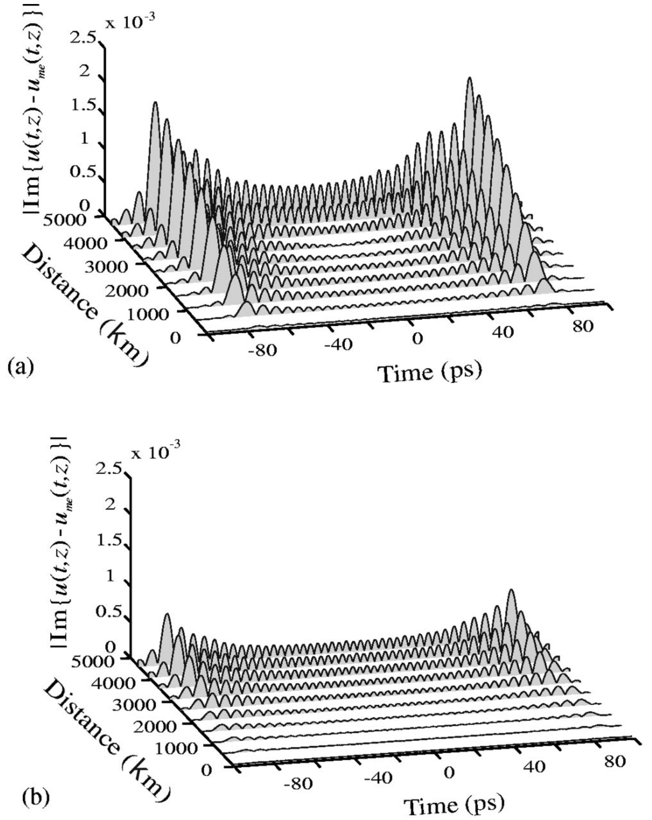

FIG. 8. Error of the imaginary part of the difference $(u-u_{me})$ as a function of the distance for the low radiating pulse of Fig. 7(a): (a) including 15 Hermite-Gauss functions in the master equation and (b) including 20.

function of *Z*. In order for the eigenvalue equation for the soliton to be fulfilled, the propagation constant of the asymptotic stationary solution obtained previously has to coincide with the highest of the H_S eigenvalues. In Fig. 5, we first observe how the eigenvalues of the Hamiltonian become constant quickly, indicating that the transient to the stationary regime is compatible with that obtained before for the λ's. And second and more important, we can clearly appreciate how the highest eigenvalue of the Hamiltonian tends to the propagation constant of the asymptotic stationary solution. This property shows that the asymptotic solution is indeed a stationary nonlinear solution of the exact evolution equation in first-order perturbation theory. In this way, after restoring the unscaled variables, we prove that the soliton propagation constant is of the form $\beta_S=2g/Lf(\tau_1)+O(g^2)$, where f is a universal function of τ_1 that can be calculated by the previous procedure.

Next, we perform an analysis of our results in terms of the number of modes, *N*, used in our Hermite-Gauss expansion. In Fig. 6, we plot the propagation constant β_S, the peak power P_0, and the root-mean-square width of the DM soliton σ, as a function of *N*. We observe that the process is clearly convergent with the number of modes. In order to confirm the convergence with *N* of the soliton solution obtained by solving the master equation (42) using the projection method, we have simultaneously performed a numerical check of the stability of the different solutions calculated with different values of *N*. First, we have obtained the soliton solutions that solve the stationary master equation for different *N*'s. Then we have used them as the initial condition of our numerical split-step Fourier method to simulate their evolution over many dispersion periods. The stability of the different solutions with *N* is apparent in Fig. 7, where we compare the evolution of the modulus of the soliton amplitude for $N=10$ and $N=2$. As expected from the convergence pattern represented in Fig. 6, the $N=10$ solution provides not only a qualitative approximation to the exact solution, but an accurate description of the DM soliton properties. The distance here considered is large enough to guarantee that the accumulated nonlinear phase $(\beta_S Z)$ is meaningful (larger than 2π). This parametric study shows the accuracy of our approach and provides a criterion to determine the adequate number of modes required for a given purpose (see also [21]).

Finally, we have also performed and alternative check of the master equation by comparing the evolution of the $N=10$ soliton solution given by the master equation (using 15 and 20 Hermite-Gauss functions) against the results provided by the split-step Fourier code for the same initial condition, i.e., those plotted in Fig. 7(a). In Fig. 8, we represent the error of the imaginary part of the difference $(u-u_{me})$ at every spatial z_j slice, where u is the pulse amplitude calculated numerically and $u_{me}(t,z_j)=\Sigma_n\lambda_n(z_j)\overline{h}(t)e^{-t^2/2}$ is the amplitude obtained by integrating the master equation (42) using 15 [Fig. 8(a)] and 20 [Fig. 8(b)] Hermite-Gauss functions. In this case, we define the error as the absolute value of these differences. Notice that the errors decrease when we increase the number of modes that we use to describe the master equation (42). In Fig. 8, we also observe that the errors grow faster for larger values of time. This is the time domain that is more sensitive to the number of modes. This fact is related to the existence of residual radiation since the soliton with $N=10$ modes is, after all, an approximation to the exact solution and, therefore, generates week dispersive waves. Dispersive waves require an increasing number of Hermite-Gauss modes to describe their evolution over long distances. A complete analogous result is obtained for the real part of the difference $(u-u_{me})$.

VI. CONCLUSIONS

We have presented a method to describe both the fast and slow dynamics of pulse evolution in strong DM systems. Nonlinearity is treated perturbatively and its effects are analytically calculated by means of a Green-function method specially adapted to DM systems. This adaptation is realized through the novel concept of proper length. The description of the pulse evolution, in both the fast and slow dynamics regimes, is ansatz-independent since no assumption about the properties of the solution has been utilized. Using this approach, we have found that a nonlinear solitonlike pulse appears as a stationary solution of the Hamiltonian of the DM system. We have studied the accuracy in the description of the soliton solution in terms of the number of Hermite-Gauss modes used in our approach finding a nice convergence pattern. Finally, the validity of all the results presented here has been explicitly checked by means of alternative numerical simulations that show the consistency of the proper length perturbation theory. We conclude that the equations obtained for describing the slow and fast dynamics of a DM system are exact in first-order perturbation theory.

ACKNOWLEDGMENTS

We are especially grateful to Professor Hermann A. Haus, who suggested this problem to us. One of us (A.F.) acknowledges his advice and kindness during his stay with Professor Haus' group. This work was financially supported by the Generalitat Valenciana (Grant No. GV98-1-79), Spain.

[1] H. A. Haus, K. Tamura, L. E. Nelson, and E. P. Ippen, IEEE J. Quantum Electron. **31**, 591 (1995); C. X. Yu, S. Namiki, and H. A. Haus, *ibid.* **33**, 649 (1997); C. X. Yu, S. Namiki, and H. A. Haus, *ibid.* **33**, 660 (1997); I. E. Nelson, D. J. Jones, K. Tamura, H. A. Haus, and E. P. Ippen, Appl. Phys. B: Lasers Opt. **65**, 277 (1997).

[2] N. Edagawa, I. Morita, M. Suzuki, S. Yamamoto, H. Taga, and S. Akiba (unpublished); N. Edagawa, S. Yamamoto, H. Taga, and S. Akiba, Electron. Lett. **31**, 2027 (1995); M. Suzuki, I. Morita, N. Edagawa, S. Yamamoto, and S. Akiba (unpublished).

[3] H. A. Haus and W. S. Wong, Rev. Mod. Phys. **68**, 423 (1996) (and references therein).

[4] V. S. Grigorian and C. R. Menyuk, Opt. Lett. **23**, 609 (1998).

[5] S. K. Turitsyn, I. Gabitov, E. W. Laedke, V. K. Mezentsev, S. L. Musher, E. G. Shapiro, T. Schäfer, and K. H. Spatschek, Opt. Commun. **151**, 117 (1998).

[6] T. Georges, J. Opt. Soc. Am. B **15**, 1553 (1998).

[7] J. N. Kutz, P. Holmes, S. G. Evangelides, Jr., and J. P. Gordon, J. Opt. Soc. Am. B **15**, 87 (1998).

[8] J. N. Kutz and S. G. Evangelides, Jr., Opt. Lett. **23**, 685 (1998).

[9] V. S. Grigorian, T. Yu, E. A. Golovchenko, C. R. Menyuk, and A. N. Pilipetskii, Opt. Lett. **22**, 1609 (1997).

[10] A. Berntson, N. J. Doran, W. Forysiak, and J. H. B. Nijhof, Opt. Lett. **23**, 900 (1998).

[11] T. I. Lakoba, J. Yang, D. J. Kaup, and B. A. Malomed, Opt. Commun. **149**, 366 (1998).

[12] S. K. Turitsyn, A. B. Aceves, C. K. R. T. Jones, and V. Zharnitski, Phys. Rev. E **58**, R48 (1998).

[13] Y. Kodoma, S. Kumar, and A. Maruta, Opt. Lett. **22**, 1689 (1997).

[14] T.-S. Yang and W. L. Kath, Opt. Lett. **22**, 985 (1997).

[15] S. K. Turitsyn, Phys. Rev. E **58**, R1256 (1998).

[16] Y. Chen and H. A. Haus, Opt. Lett. **23**, 1013 (1998).

[17] T. I. Lakoba and D. J. Kaup, Electron. Lett. **34**, 1124 (1998).

[18] J. C. Bronski and J. N. Kutz, Physica D **108**, 315 (1997).

[19] G. P. Agrawal, *Nonlinear Fiber Optics* (Academic Press, San Diego, 1995).

[20] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, San Diego, 1980).

[21] S. K. Turitsyn and E. G. Shapiro, J. Opt. Soc. Am. B **16**, 1321 (1999).